LECTURE 4: NON-UNIFORM SAMPLING, FUNCTIONAL ANALYSIS

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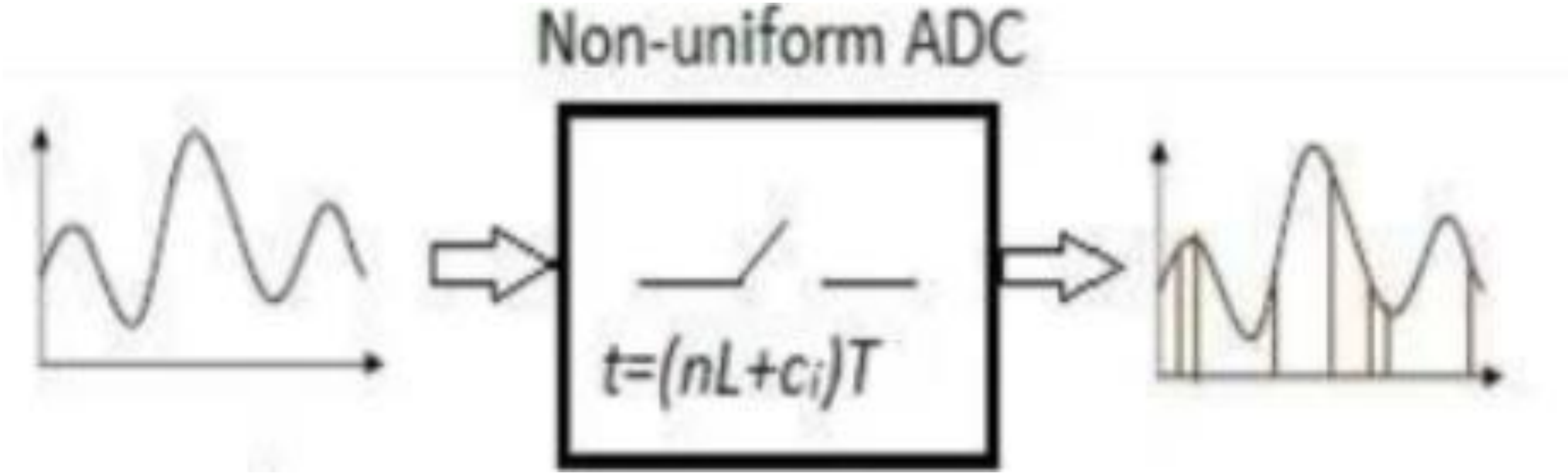


NON-UNIFORM SAMPLING

לעיתים נדרשת דגימה לא סופית, כלומר
המרווח בין הדגימות אינו אחיד.

* תהליכים אקראיים לאורך הזמן מפוזרים לא באופן אחיד
* דגימה מהרכבות מעין מיקרופונים בעולם לכן!

* מידע מעודכנים לגבי הסביבה.

WHAT IS NON-UNIFORM SAMPLING (NUS)?

- The general theory for non-baseband and nonuniform samples was developed in 1967 by Henry Landau. He proved that the average sampling rate (uniform or otherwise) must be twice the occupied bandwidth of the signal, assuming it is a priori known what portion of the spectrum was occupied.
- In the late 1990s, this work was partially extended to cover signals for which the amount of occupied bandwidth was known, but the actual occupied portion of the spectrum was unknown.
- In the 2000s, a complete theory was developed using compressed sensing. In particular, the theory, using signal processing language, is described in that 2009 paper. They show, among other things, that if the frequency locations are unknown, then it is necessary to sample at least at twice the Nyquist criteria; in other words, you must pay at least a factor of 2 for not knowing the location of the spectrum. Note that minimum sampling requirements do not necessarily guarantee numerical stability.

WHAT IS NON-UNIFORM SAMPLING?

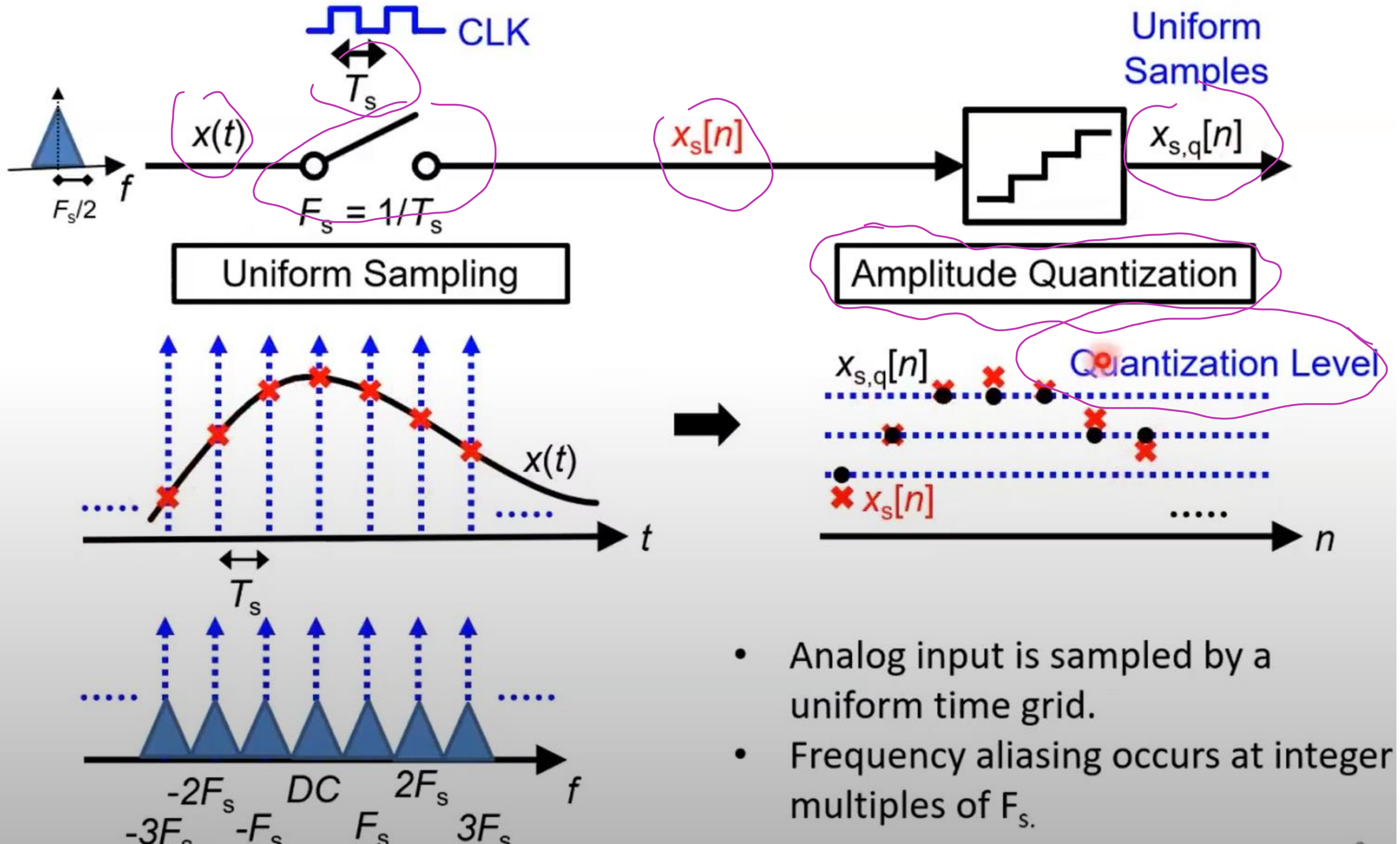
- Nonuniform sampling is a branch of sampling theory involving results related to the Nyquist–Shannon sampling theorem. Nonuniform sampling is based on Lagrange interpolation and the relationship between itself and the (uniform) sampling theorem. Nonuniform sampling is a generalization of the Whittaker–Shannon–Kotelnikov (WSK) sampling theorem.
- The sampling theory of Shannon can be generalized for the case of nonuniform samples, that is, samples not taken equally spaced in time. The Shannon sampling theory for non-uniform sampling states that a band-limited signal can be perfectly reconstructed from its samples if the average sampling rate satisfies the Nyquist condition. Therefore, although uniformly spaced samples may result in easier reconstruction algorithms, it is not a necessary condition for perfect reconstruction.

WHAT IS NON-UNIFORM SAMPLING?

- Uniform sampling for which the interval between samples is uniform and fulfills: $t_n = nD$ with n-integer and D is the sampling time.
- **Non-uniform** sampling meaning that the interval between the samples is not uniform.
- Examples:

1) Spatial sampling with array of microphones.
2) Information from cellular phones on the surrounding.

Most ADCs we used today…



- Analog input is sampled by a uniform time grid.
- Frequency aliasing occurs at integer multiples of F_s.

Sampling Theorem Generalization

On Nonuniform Sampling of Bandwidth-Limited Signals*

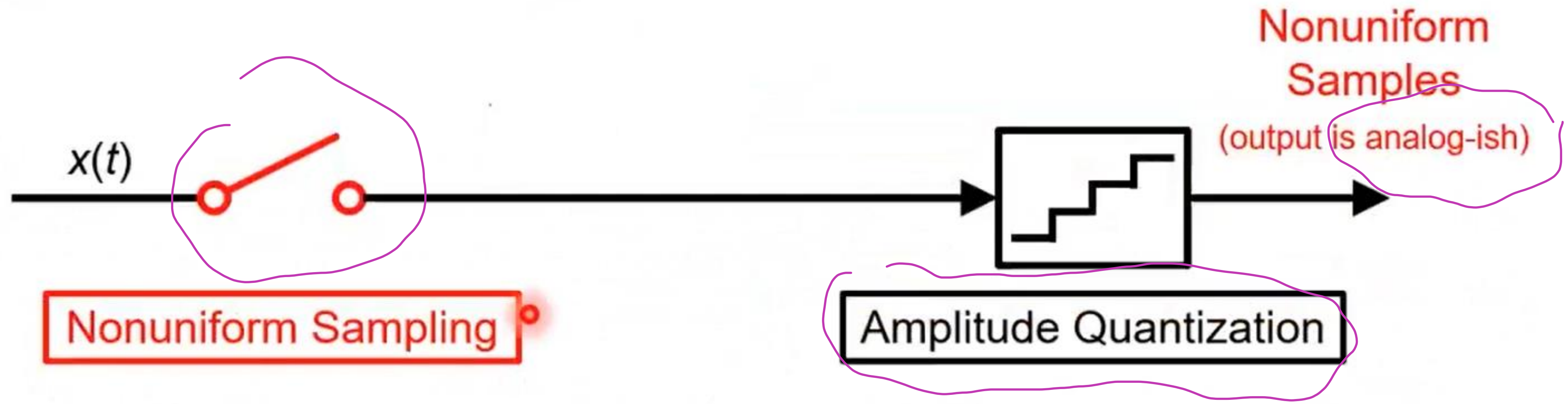
J. L. YEN†

FOR A bandwidth-limited signal whose Fourier spectrum contains no component above frequency W cycles per second the well-known sampling theorem holds. The signal is uniquely determined by its values at an infinite set of sample points spaced at $1/2w$ seconds apart. Although the sampling theorem came into prominence as a tool of interpolation after Whittaker,[1,2] and later on rediscovered as an important aspect of communication theory by Nyquist,[3] Gabor,[4] and Shannon,[5] it was already known to Cauchy.[6] In fact, Black[7] attributed the following statement to Cauchy: "If a signal is a magnitude-time function, and if time is divided into equal intervals such that each subdivision comprises an interval T seconds long where T is less than half the period of the highest significant frequency component of the signal, and if one instantaneous sample is taken from each subinterval in any manner; then a knowledge of the instantaneous magnitude of each sample plus a knowledge of the instant within each subinterval at which the sample is taken contains all of the information of the original signal." Although the statement is still not precise enough, it includes the idea of nonuniform sampling. Black also discussed the use of nonuniform sampling with the aid of some examples, and stated that their application can at most cause some loss in accuracy and in simplicity of the reconstruction procedure. However, for a more thorough understanding of nonuniform sampling, it is necessary not only to prove the unique specification of signals but also to find the explicit reconstruction formulas involved. A knowledge of the latter provides simple estimates of complexity and accuracy required thus enabling one to ascertain whether a particular nonuniform sampling process has any practical

- So long as average sample rate meets Nyquist rate, there is no loss of info.

[1] J.L. Yen, "On Nonuniform Sampling of Bandwidth-Limited Signals," IRE Transactions on Circuit Theory, 1956
[2] H. S. Black, "Modulation Theory, " D. Van Nostrand Cd.; Inc., New York, N. Y., 1953.

Sampling input at irregular time?

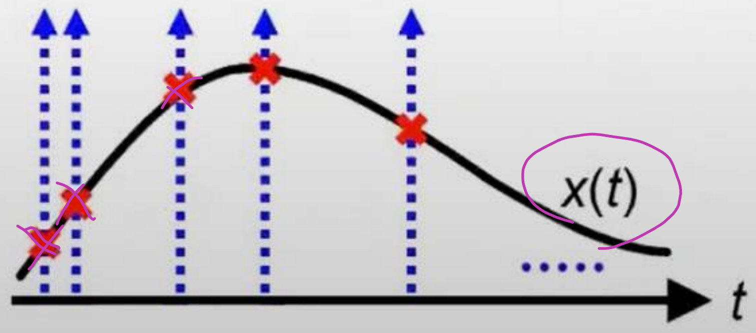


- Analog input is sampled by nonuniform time grid.
- Different types of NUS:

  random sampling,

  multi-coset sampling,

  level-crossing sampling, …



- Aliasing pattern changes ➔ signal-dependent sampling is alias free.

MATLAB EXAMPLE

```
% Non uniform sampling example
%
% Five climate stations are placed along a road of length 1000km.
% x0 are the positions, g0 are temperature measurements
% Approximate the temperature along the road

close all;
clear all;
clc;

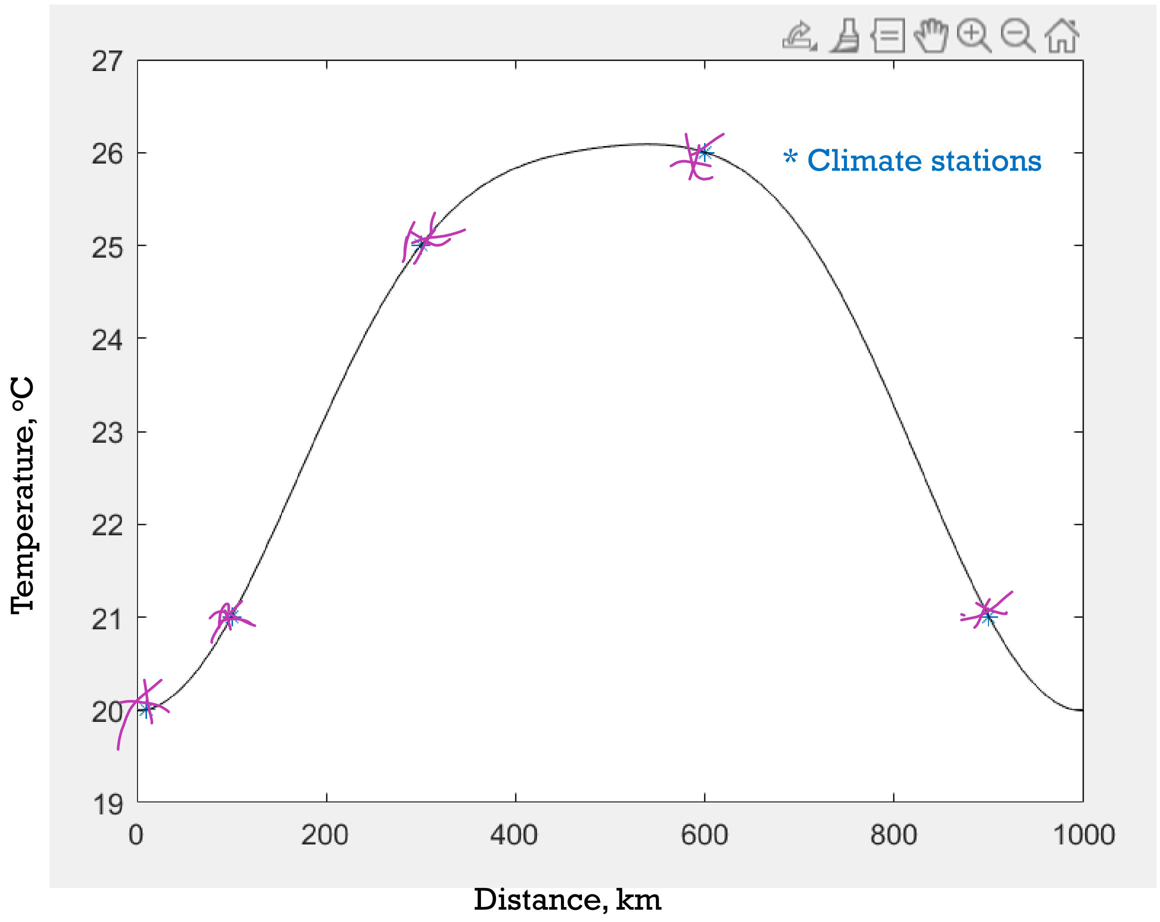
% problem input data
L=1000; % road length
x0=[10 100 300 600 900]'; % positions of stations
g0=[20 21 25 26 21]'; % temperature at stations
% try to change g0(5)=21 to g0(5)=25, ... explanation?
k=[-2:2]; % indexes of Fourier coefficients
w0=2*pi/L; % fundamental frequency

% compute series
F0=exp(j*x0*k*w0); a0=inv(F0)*g0;

% optional analysis
% rank(F), det(F), sort(abs(eig(F0))),

% reconstruction with high resolution
x=linspace(0,L,256)’;
F=exp(j*x*k*w0);
g=F*a0;

% plot
figure;
plot(x0,g0,'*',x,g,'k-');
```



ASSUMPTIONS

- **Band-limited signal** - otherwise cannot be reconstructed
- Periodic – to facilitate the calculations; one period is defined on finite length and therefore easier to analyze the sampling.
- There is a Fourier series with finite number of coefficients.
- Let $x(t)$ be the periodic signal with period T
- a_k Fourier series coefficients fulfill $a_k = 0 \quad \forall|k| > M$

therefore, we can write $x(t_n) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t_n}$, $\omega_0 = \frac{2\pi}{T}$

Now we will sample the signal $x(t)$ with N samples in one period T but with time intervals $t_n \in [0, T]$, $0 \le n \le N-1$ not uniform

therefore, we can write

$$x(t_n) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t_n}, \qquad 0 \le n \le N-1$$

The goal is to reconstruct $x(t)$ and the question is how?

RECONSTRUCTION

- Reconstruction from the Fourier coefficients a_k
- Note the recent equation is simply system of **N** equations that can be rewritten as:

$$x(t_0) = a_{-M}e^{-jM\omega_0 t_0} + \cdots + a_M e^{jM\omega_0 t_0}$$

$$\vdots$$

$$\vdots$$

$$x(t_{N-1}) = a_{-M}e^{-jM\omega_0 t_{N-1}} + \cdots + a_M e^{jM\omega_0 t_{N-1}}$$

this is the system of N equations with $2M + 1$ variables a_k. To reconstruct $x(t)$ from $x(t_n)$, we will solve the system of equations. So, we basically have a problem in linear algebra instead of the sampling and reconstruction problem in signal processing.

- We write the systems of equations in matrix form as:

$$\underline{x} = [x(t_0), x(t_1), \ldots, x(t_{N-1})]^T$$

We want vector column not as row

this is the sampled vector of size Nx1

$$\underline{a} = [a_{-M}, a_{-M+1}, \ldots, a_0, \ldots, a_{M-1}, a_M]^T$$

this is a vector of Fourier series coefficients of size 2M+1x1

RECONSTRUCTION

- Now we will express the exponents of Fourier series in matrix form as the following Fourier exponents matrix is of size Nx2M+1:

- $$\underline{\underline{F}} = \begin{bmatrix} e^{-jM\omega_0 t_0} & \cdots & e^{-j1\omega_0 t_0} & e^{-j0} & e^{j1\omega_0 t_0} & \cdots & e^{jM\omega_0 t_0} \\ e^{-jM\omega_0 t_1} & \cdots & e^{-j1\omega_0 t_1} & e^{-j0} & e^{j1\omega_0 t_1} & \cdots & e^{jM\omega_0 t_1} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ e^{-jM\omega_0 t_{N-1}} & \cdots & e^{-j1\omega_0 t_{N-1}} & e^{-j0} & e^{j1\omega_0 t_{N-1}} & \cdots & e^{jM\omega_0 t_{N-1}} \end{bmatrix}$$

Columns repeat themselves with respect to **k**
Rows change with respect to **n** which increases

This system of equations can be written as $\underline{x} = \underline{\underline{F}}\,\underline{a}$

With $\underline{x}$ samples, M coefficients and t_n samplings time

Question: What should be given in order to write the equation and find $\underline{a}$?

$F \to \omega_0, 0 \leq n \leq N-1, t_n, M$

$\underline{x} \to$ samples

RECONSTRUCTION

- Solution:

1. $N = 2M + 1$, $\underline{\underline{F}}$ is square matrix, solution is $\underline{a} = \underline{\underline{F}}^{-1} \cdot \underline{x}$

- Let $\underline{\underline{F}}$ has an inverse, $\text{rank}\left(\underline{\underline{F}}\right) = N$; $\det\left(\underline{\underline{F}}\right) \neq 0$, rows/columns are independent

2. $N < 2M + 1$: more variables than equations

- There are ∞ solutions, therefore we won't be able to obtain the correct solution (reconstruction).

than variables.

3. $N > 2M + 1$: more ~~variables than~~ equations – in general will be either one or none solutions. In our case, if the model is correct, there is should be one solution.

RECONSTRUCTION

- <u>One option</u>: remove equations and solve square system (מערכת ריבועית)
- <u>Second option</u>: if there is a noise or inaccuracy, the square system solution will not be accurate. Therefore, instead of $\underline{x} = \underline{\underline{F}}\,\underline{a}$

we will solve a minimization problem with the goal to bring to the minimum $\underline{a}$

$$\min_{\underline{a}} ||\underline{x} - \underline{\underline{F}}\,\underline{a}||_2$$

The solution here is $\underline{a} = \underline{\underline{F}}^T \underline{x}$, $\underline{\underline{F}}^T = \left(\underline{\underline{F}}^H \underline{\underline{F}}\right)^{-1} \underline{\underline{F}}^H$ and named pseudo-inverse

due norm 2 the information is more immune to noise.

Note: the more rows we will have the more precise will be the solution.

H – Hermitian transpose = complex conjugate and transpose

RECONSTRUCTION

4. In case of square matrix,

$\underline{\underline{F}}$ has a structure of Vandermonde matrix:

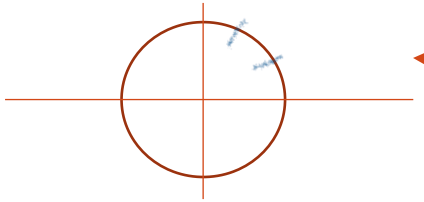
$$\underline{\underline{F}} = \begin{pmatrix} {C_0}^1 & {C_0}^2 & \cdots & {C_0}^{2M+1} \\ \vdots & & & \\ {C_{N-1}}^1 & {C_{N-1}}^2 & \cdots & {C_{N-1}}^{2M+1} \end{pmatrix}$$

Home task to continue

$$\det\left(\underline{\underline{F}}\right) = \overbrace{\underbrace{\prod}_{n>n'}}^{multiplicaiton} (C_n - C_{n'})$$ - properties of Vandermonde matrix (the determinant of a square Vandermonde matrix)

So $\det \neq 0$ in case $C_n \neq C'_n$ but $C_n \propto e^{j\omega_0 t_n} = e^{j\frac{2\pi}{T}t_n}$ $\quad t_n \in [0,2\pi)$

$\omega_0 = \frac{2\pi}{T}$



C_n are points on unit circle

Now, for different t_n we will obtain different C_n. Therefore, we can choose different t_n because $\underline{\underline{F}}$ is reversible and therefore the solution will exist.

Therefore $\det \neq 0$ in case $t_n \neq t'_n$ a strong solution meaning the reconstruction is assured!

ALEXANDRE-THÉOPHILE VANDERMONDE
(28 FEBRUARY 1735 – 1 JANUARY 1796)

- Vandermonde was a violinist, and became engaged with mathematics only around 1770
- French mathematician
- He was professor at the École Normale

Supérieure, member of the Conservatoire

national des arts et métiers and examiner

at the École polytechnique

RECONSTRUCTION

5. When rows/columns are dependent $\det \neq 0$ or eigenvalue $\lambda_i = 0$

What happens when the rows are "almost dependent"?

λ_i is very small $\lambda_i \approx 0$ which his not good in practice.

In practice, there will be a numerical issue to calculate the inverse matrix when $\lambda_i \approx 0$

Let explore the ratio between eigenvalues

$$\text{cond}(\underline{\underline{F}}) = \frac{|\lambda_{max}|}{|\lambda_{min}|}$$

מספר המצב

State number

In practice we will request: cond->1

$\underline{\underline{F}}$ is normalized

$1 \leq \text{cond}(\underline{\underline{F}}) \leq \infty$

RECONSTRUCTION

$\underline{\underline{F}}$ is normalized

$1 \leq \mathrm{cond}(\underline{\underline{F}}) \leq \infty$

$\underline{\underline{A}}\ \underline{x} = \underline{b}$ the meaning is amplification of the noise, and so

$\underline{x} = \underline{\underline{A}}^{-1}\ \underline{b}$

$\underline{x} = \underline{\underline{A}}^{-1}\underline{b} + \underline{\underline{A}}^{-1}e$

$$\frac{||\underline{\underline{A}}^{-1}e||_2/||A^{-1}\underline{\underline{b}}||_2}{||e||_2/||b||_2} = \mathrm{cond}(\underline{\underline{A}})$$

MATLAB EXAMPLE

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% rank(F), det(F), sort(abs(eig(F0))),

% reconstruction with high resolution
x=linspace(0,L,256)';
F=exp(j*x*k*w0);
g=F*a0;

% plot
figure;
plot(x0,g0,'*',x,g,'k-');
```

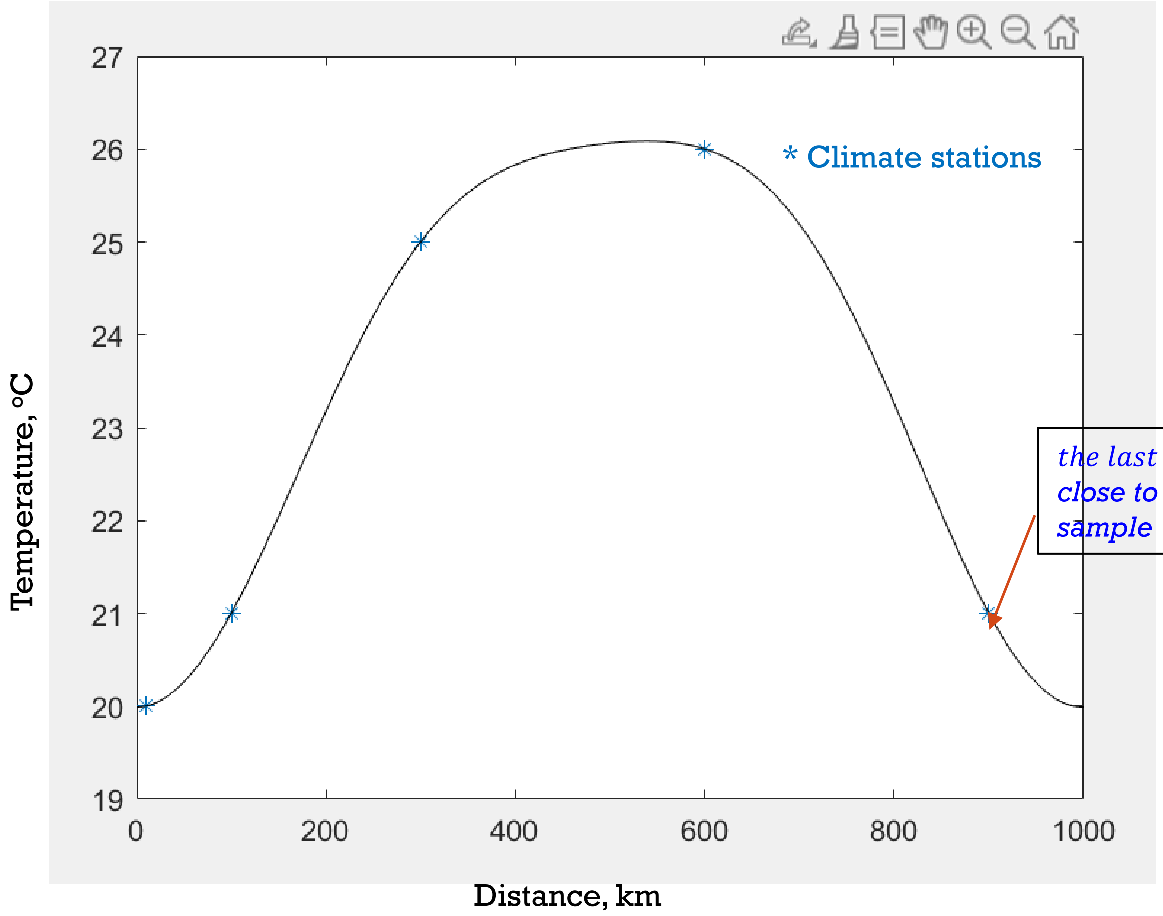
```
w0 =

    0.0063
```

Furier coefficients

```
a0 =

  -0.3125 + 0.0223i
  -1.5188 - 0.0361i
  23.6507 + 0.0000i
  -1.5188 + 0.0361i
  -0.3125 - 0.0223i
```



the last sample is close to the initial sample

Looks reasonabe

MATLAB EXAMPLE: CHANGE OF ONE SAMPLE

```
% Non uniform sampling example
%
% Five climate stations are placed along a road of length 1000km.
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% Approximate the temperature along the road

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clc;

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% rank(F), det(F), sort(abs(eig(F0))),

% reconstruction with high resolution
x=linspace(0,L,256)';
F=exp(j*x*k*w0);
g=F*a0;

% plot
figure;
plot(x0,g0,'*',x,g,'k-');
```

```
w0 =

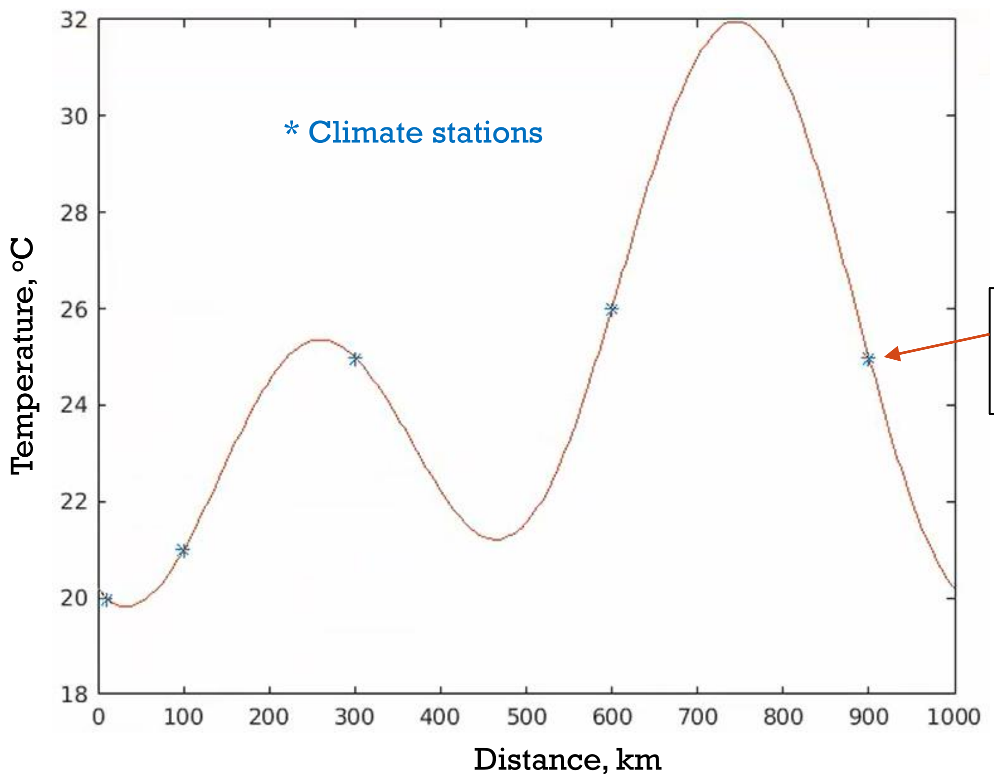
    0.0063
```

Furier coefficients

```
>> cond(F0)

ans =

    6.3382
```



the last sample is too far from the initial sample

Doeas not look reasonabe

SAMPLING AND FUNCTIONAL ANALYSIS

WHAT IS FUNCTIONAL ANALYSIS?

- Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear functions defined on these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformations of functions such as the Fourier transform as transformations defining continuous, unitary etc. operators between function spaces

RECALL EIGENVECTORS

Let assume functions $f(x)$ and $g(x)$ and defined by inner product $\mathcal{L}_2(\mathbb{R}), x \in \mathbb{R}$

$\mathcal{L}_2$-norm

f, g square integrable

$$\text{Inner product: } < f, g >= \int_{-\infty}^{\infty} f(x) g^*(x)\, dx$$

Given collection of basis functions $\phi_n(x)$ which are orthogonal and span the sub-space V in $\mathcal{L}_2(\mathbb{R})$

Projection of $f(x)$ on V will take place by description of f by ϕ_n: projection

$$\hat{f}_x = \sum_{n=-\infty}^{\infty} C_n \phi_n(x)$$

We will find the coefficients by activating the inner product

so $C_n =< f, \phi_n >$ while ϕ_n is orthogonal

$C_n =< f, \phi_n > / < \phi_n, \phi_m >$ while $||\phi_n||^2$ is orthonormal

otherwise, the relation between C_n and $< f, \phi_n >$ is given by the mapping $< \phi_n, \phi_m >$

RELATION OF EIGENVECTORS TO SAMPLING

Let choose $\phi_n(t) = \text{sinc}\left(\frac{t-nT}{T}\right) \quad t \in \mathbb{R}, n \in \mathbb{Z}$

1. Are $\phi_n(t)$ orthogonal?
2. Which sub-space they span?
3. What is the meaning of the projection of $f(t)$ on this space?

RELATION OF EIGENVECTORS TO SAMPLING

1. Orthogonal

by Parseval theorem

$< \phi_n(t), \phi_m(t) >=, \ t \in R, \ m,n \in Z$

$$= \int_{-\infty}^{\infty} sinc\left(\frac{t-nT}{T}\right) sinc\left(\frac{t-mT}{T}\right) \overset{FT}{dt} = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} T \cdot \Pi\left(\frac{\omega}{\omega_s}\right) e^{-j\omega nT} \cdot e^{j\omega t} \cdot d\omega \cdot sinc\left(\frac{t-mT}{T}\right) dt =$$

נחליף סדר

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} T\Pi\left(\frac{\omega}{\omega_s}\right) e^{-j\omega nT} \underbrace{\int_{-\infty}^{\infty} sinc\left(\frac{t-mT}{T}\right) e^{j\omega t}\, dt}_{FT-\omega}\, d\omega =$$

התמרה ω

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} T\Pi\left(\frac{\omega}{\omega_s}\right) e^{-j\omega nT} \cdot T\Pi\left(\frac{-\omega}{\omega_s}\right) e^{+j\omega mT}\, d\omega$$

פונקציה זוגית

$$= T \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} T\Pi\left(\frac{\omega}{\omega_s}\right) e^{-j\omega \underbrace{(m-n)T}_{t \text{ of } IFT}}\, d\omega = T \cdot sinc\left(\frac{\overbrace{[m-n]T}^{t \text{ of } IFT}}{T}\right) = T \cdot \delta[m-n]$$

Therefore, ϕ_n are orthogonal functions which span band-limited sub-space

RELATION OF EIGENVECTORS TO SAMPLING

2. Which sub-space they span?

they span the sub-space of band-limited functions $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ meaning

$$\mathrm{V} \equiv \mathcal{L}_2\left(\left[-\frac{\pi}{T}, \frac{\pi}{T}\right]\right)$$

RELATION OF EIGENVECTORS TO SAMPLING

3. What is the meaning of the projection of $f(t)$ on this space?

The meaning of the projection is: <u>representation of f by basis functions</u>

Let assume $f \in V$ and band-limited.

We aim to present $f(t)$ as $\hat{f}_x = \sum_{n=-\infty}^{\infty} C_n \phi_n(x)$, and so $f(t) = \sum_{n=-\infty}^{\infty} C_n \text{sinc}\left(\frac{t-nT}{T}\right)$

from Shannon reconstruction formula: $f(t) = \sum_{n=-\infty}^{\infty} f(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$

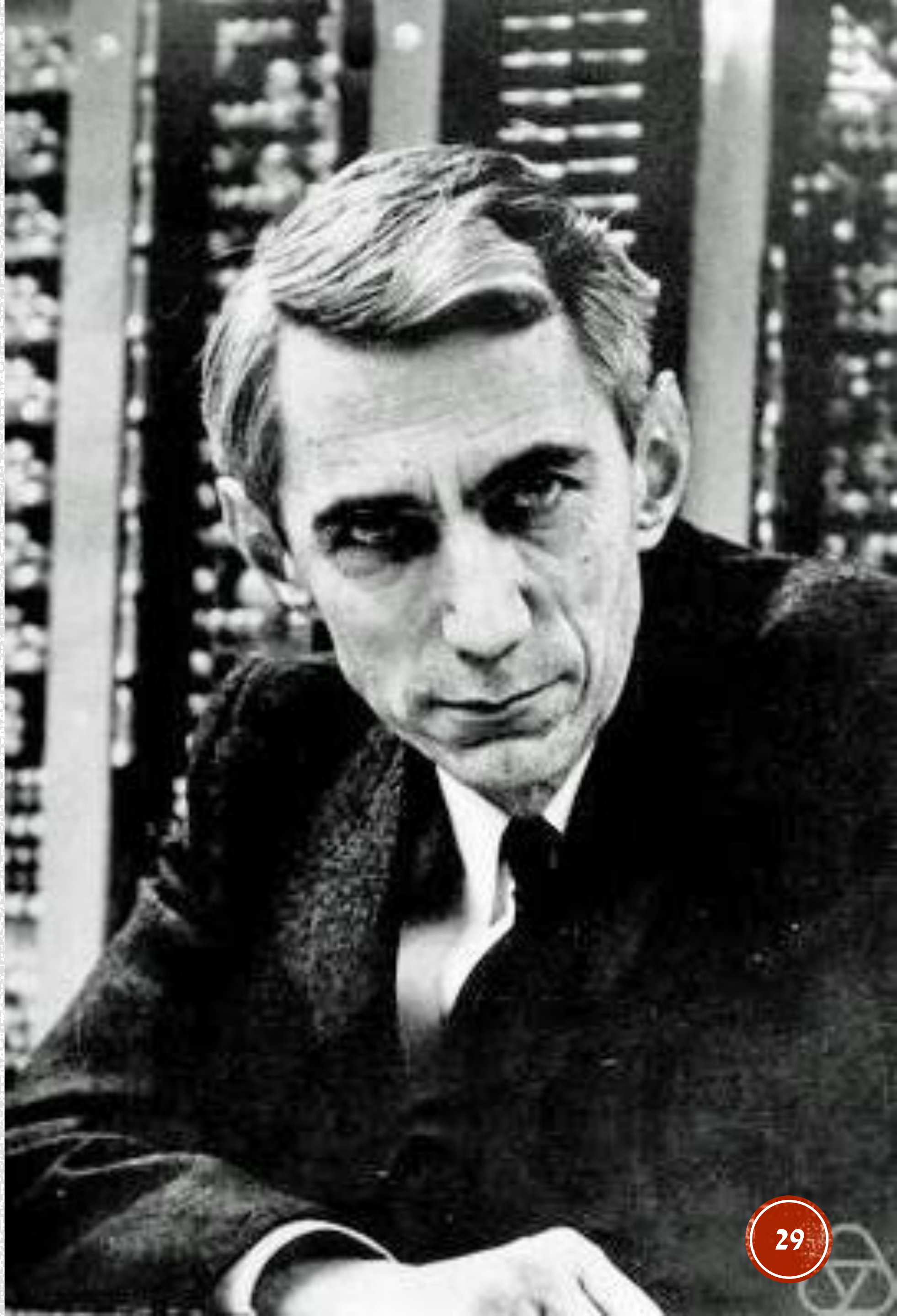
and therefore, $C_n = f(nT)$, from here also $< f, \phi_n >= C_n < \phi_n(t), \phi_m(t) \geq Tf(nT)$

so we can develop the following expression $\int_{-\infty}^{\infty} f(t)\text{sinc}\left(\frac{t-nT}{T}\right) dt = T \cdot f(nT)$

One way to calculate the samples is to <u>multiply by sinc</u> and calculate the integral – the result of the integral depends only on f(nT)

CLAUDE ELWOOD SHANNON (APRIL 30, 1916 – FEBRUARY 24, 2001)

- an American mathematician, electrical engineer, and cryptographer known as a "father of information theory".
- 21-year-old master's degree student at the Massachusetts Institute of Technology (MIT), he wrote his thesis demonstrating that electrical applications of Boolean algebra could construct any logical numerical relationship.
- Shannon contributed to the field of cryptanalysis for national defense of the United States during World War II, including his fundamental work on codebreaking and secure telecommunications.
- Using the property of electrical switches to implement logic is the fundamental concept that underlies all electronic digital computers. Shannon's work became the foundation of digital circuit design, as it became widely known in the electrical engineering community during and after World War II.



SUMMARY

Sampling each T seconds and reconstruction by function sinc is like:

- The projection of $f(t)$ on sub-space of band-limited signals $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ in $\mathcal{L}_2(\mathbb{R})$
- The coefficients of the projection are the samples of $f(nT)$
- Basis functions are $\operatorname{sinc}\left(\frac{t-nT}{T}\right)$
- Question: are $\phi_n = \operatorname{sinc}\left(\frac{t-nT}{T}\right)$ span sub-space? Yes, from the ideal sampling (Nyquist and) ideal reconstruction (Shannon)

EXAMPLE

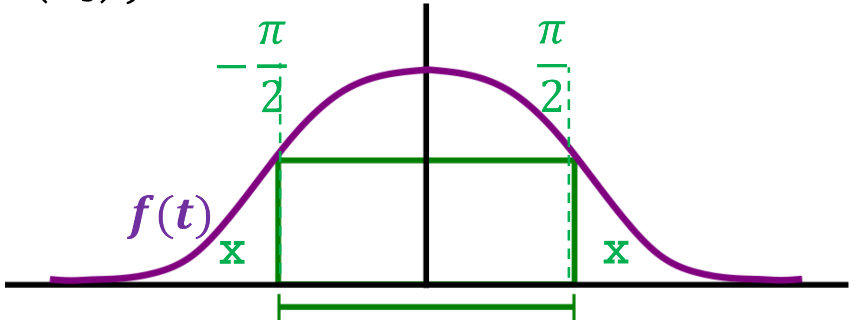
- Assume $f \in \mathcal{L}_2(\mathbb{R})$ but $f \notin V$ and therefore is not band-limited. Calculate projection coefficients of f on V and therefore is not band-limited.
- Calculate projection of f on V

1) Mathematical solution: $C_n = < f, \phi_n > / < \phi_n, \phi_m > \neq f(nT)$ are not equal to samples $C_n \neq f(nT)$

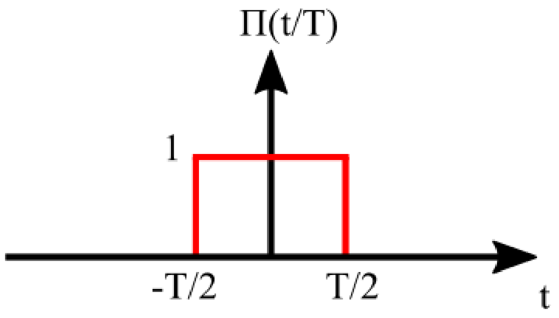
2) Signals processing solution: the projection of f on V is just a LPF:

$$\hat{f}(t) = \mathcal{F}^{-1}\left\{F(j\omega) \cdot \Pi\left(\frac{\omega}{\omega_s}\right)\right\}$$ and so

$C_n = \hat{f}(nT)$



RECTANGLE (SQUARE PULSE) AS A BASIS FUNCTION

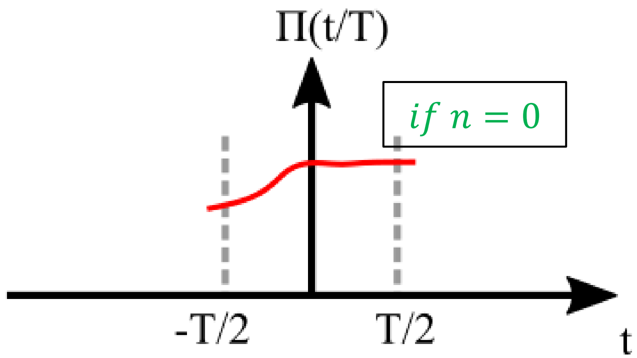


shifted each nT

width T sec

- Let choose basis function defined as $\phi_n(t) = \Pi\left(\frac{t-nT}{T}\right)$
- Is it orthogonal? Yes - check inner product. The function is not overlapping in time.
- Window of width T, is shifted in T seconds: $\int_{-\infty}^{\infty} \Pi\left(\frac{t-nT}{T}\right) \cdot \Pi\left(\frac{t-mT}{T}\right) \, \mathrm{d}t = T\delta[n-m]$
- The meaning is the projection of signal f on the sub-space which is defined by ϕ_n: This is the sub-space of staircase-signals of width T.
- What is C_n?

$$C_n = < f, \phi_n > / T = \frac{1}{T} \int_{nT-T/2}^{nT+T/2} f(t) 1 \, \mathrm{d}t$$



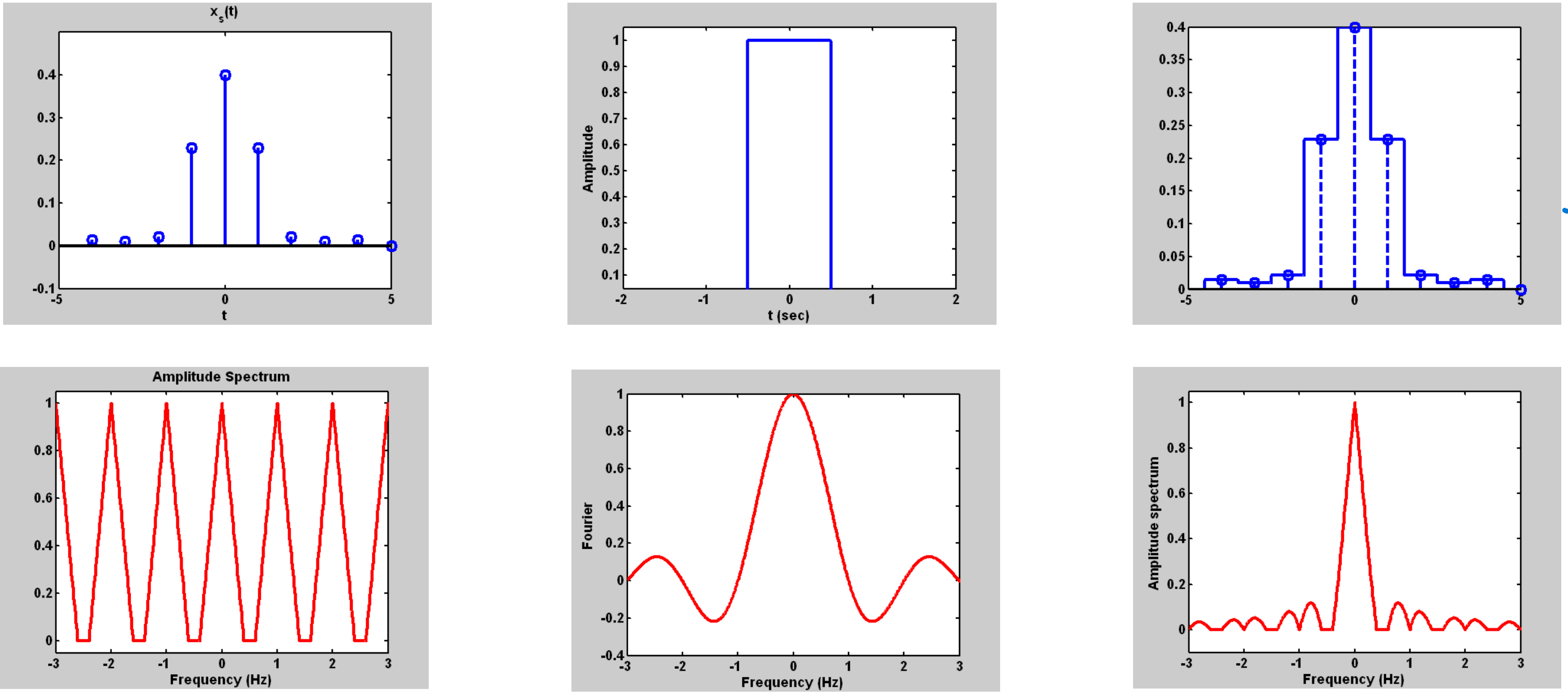
This is the average of the signal in a region of the window.

Important: this is different from the sampling with zero-order hold(ZoH)!

Assuming C_n how do we reconstruct ? $\hat{f}(t) = \sum_{n=-\infty}^{\infty} C_n \Pi\left(\frac{t-nT}{T}\right)$ – similar to ZoH

INTERPOLATION MEANS MULTIPLICATION OF SPECTRA

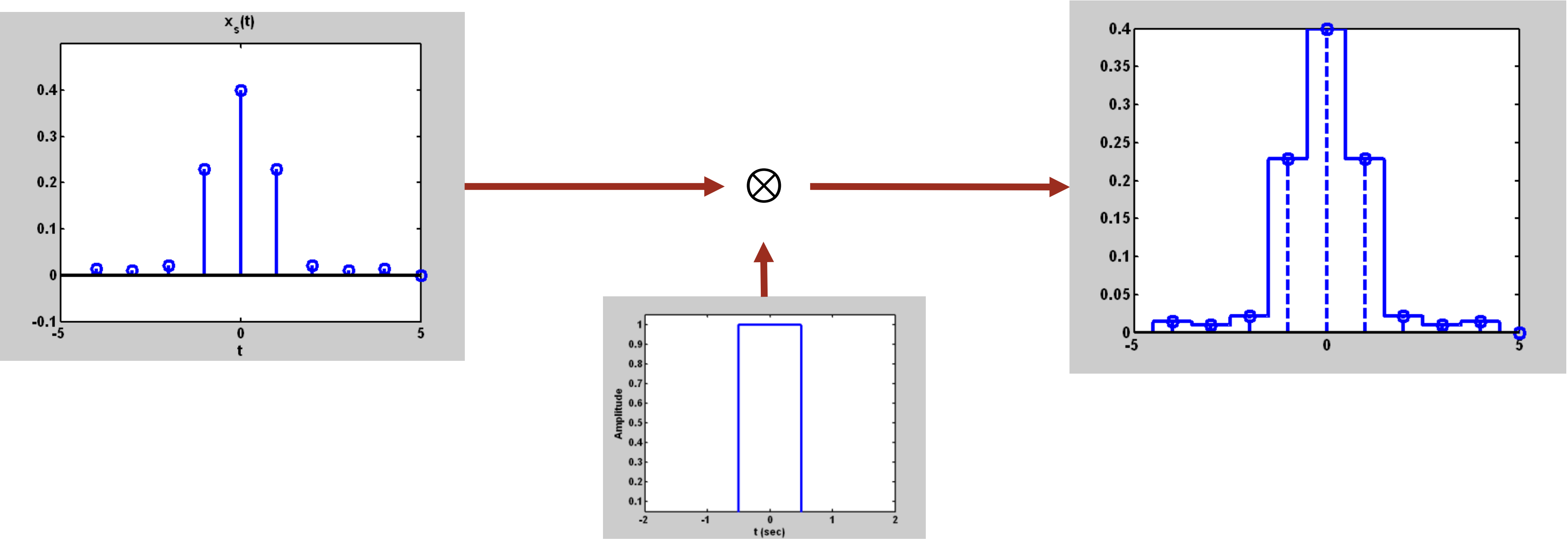
- Interpolating with the rectangular window is like multiplying by its spectrum



ZERO-HOLD INTERPOLATION TO RECONSTRUCT A SAMPLED SIGNAL

- We could simply interpolate with a rectangular window
  - This means convolving with a **square pulse**
  - So we are applying a filter

Filter's impulse response: $h(t) = \mathrm{rect}\left(\frac{t}{T}\right)$



SURVEY: FUNCTIONAL ANALYSIS

- EasyPolls:

f(t) is projected into V defined by shifted rectangular windows of width T. Can f(t) be reconstructed from Cn precisely?

○ No, the window functions are not orthogonal

○ Yes, the window functions does form an orthogonal basis

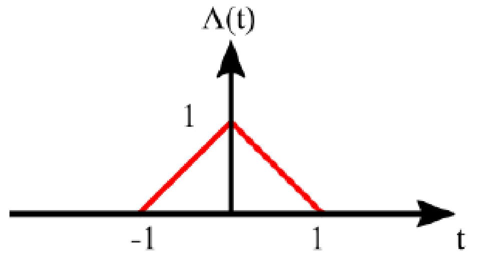
✓ ○ Only if f(t) belongs to V

results | vote

תשובה:
רק אם f שייכת ל V
שאז f היא
בסיס אורתוגונלי ני
לשחזר f(t) במלואה
לפי Cn ~
ההגדרה.

f שכל ∈ V
שמצב ...
תלויות הדדים
באורך T.
אם ניתן לשחזר f(t)
במלואה מ Cn ~
?

TRIANGLE AS A BASIS FUNCTION



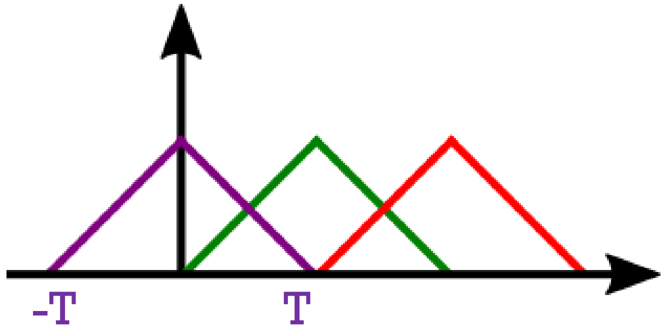
- Let choose basis function defined as $\phi_n(t) = \Lambda\left(\frac{t-nT}{T}\right)$

Orthogonality check : inner product: $\int_{-\infty}^{\infty} \Lambda\left(\frac{t-nT}{T}\right) \cdot \Lambda\left(\frac{t-mT}{T}\right)\, dt =$

- Is it orthogonal? No:

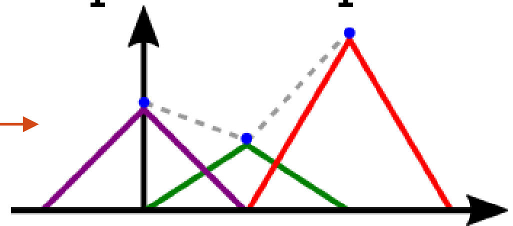
$$= \begin{cases} 2\frac{T^3}{3}, & m = n \\ \neq 0, & |m-n| = 1 \\ 0, & |m-n| > 1 \end{cases}$$

$$\int_0^{T} t \cdot t\, dt = \int_0^{T} t^2\, dt = \frac{t^3}{3}\Big|_0^T = \frac{T^3}{3}$$



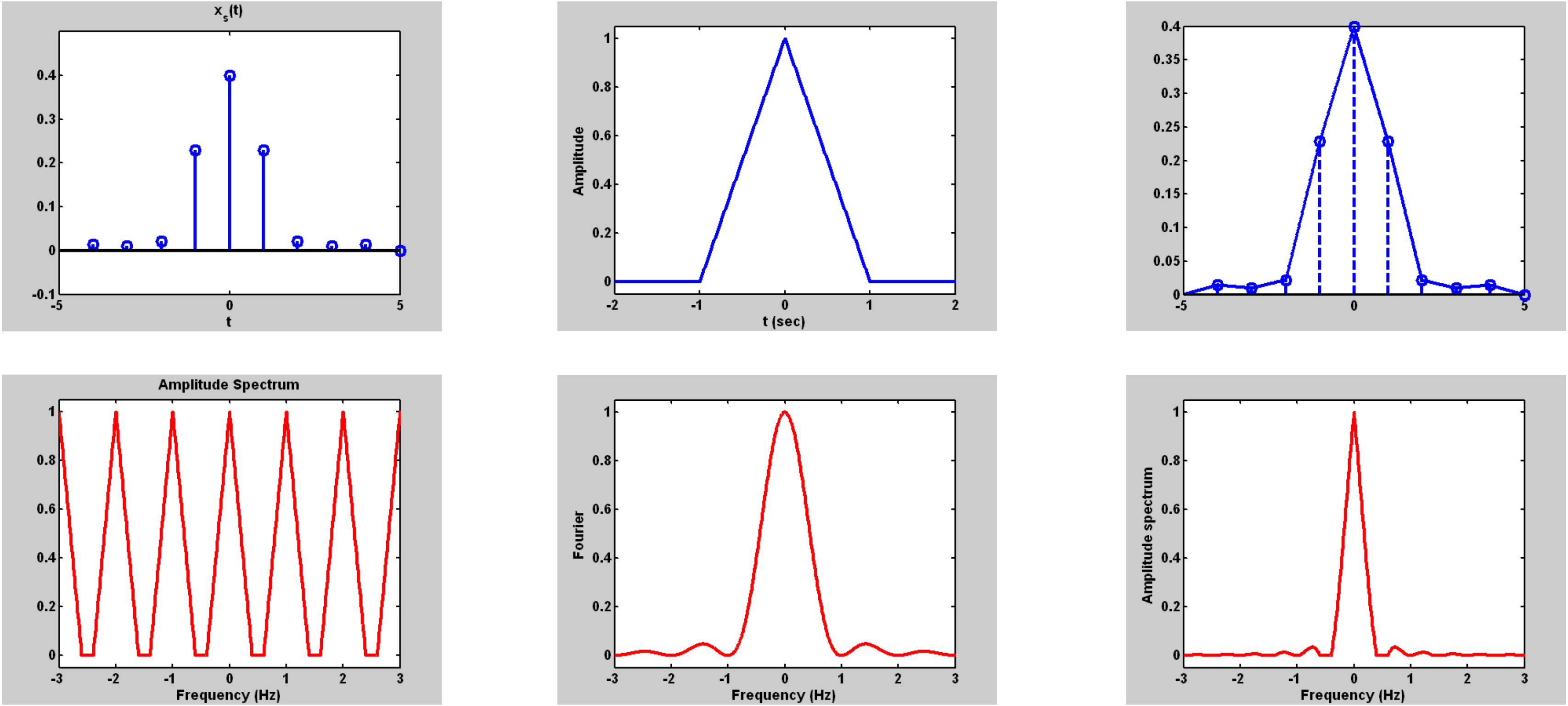
- Independent? Yes, because the tip of the triangle cannot be represented via other triangle functions $\phi_n(t)$ – all of them are zero at the tip.
- Time-invariant? Yes because the tip of the triangle cannot be represented via linear combination of other triangles.
- Which space do they span? Space of functions built from the points in space T connected via straight lines: linear interpolation.

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} C_n \Lambda\left(\frac{t-nT}{T}\right)$$



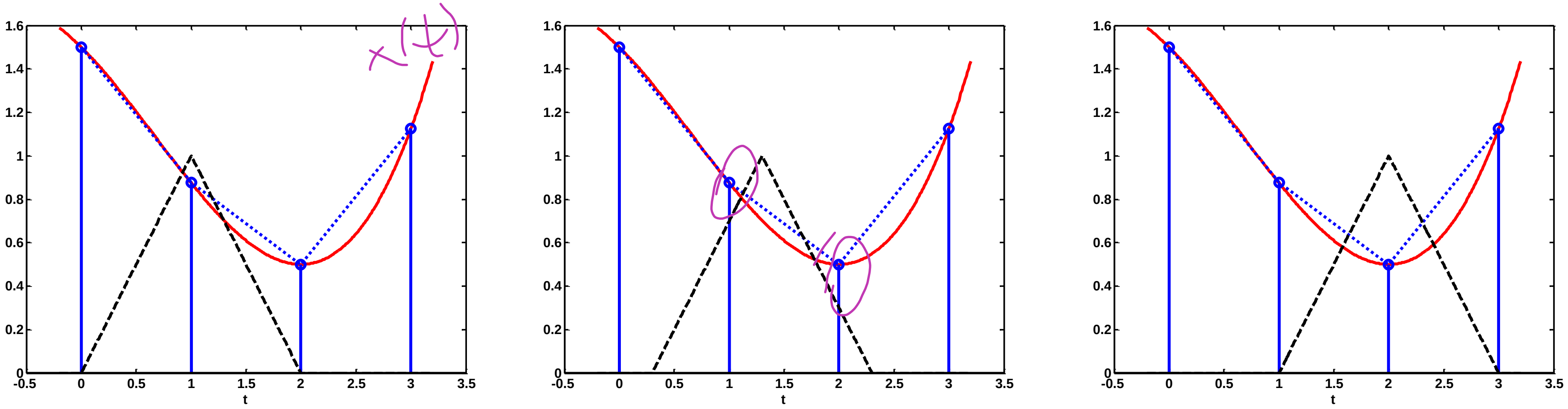
LINEAR INTERPOLATION GIVES A BETTER APPROXIMATION

- Interpolating with the triangle is like multiplying by its spectrum



WHAT DOES CONVOLUTION WITH A TRIANGLE LOOK LIKE?

- When centered on a sample point
  - No contribution from adjacent points
- When between two points
  - Sum of contributions from each point is one
  - Sum is weighted by proximity to each point



RECONSTRUCTION: GENERAL REPRESENTATION

- Assuming samples $x(nT)$, one can reconstruct (not necessarily precisely) the $x(t)$:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(nT)h(t - nT)$$

- When $h(t)$ is reconstruction function or $x(t)$ is band-limited in range $\left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$
- $h(t) = \text{sinc}(t/T)$ will give the full/ideal reconstruction but requires ∞ samples
- $h(t) = \Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ ZoH, requires 1 sample for each calculation
- $h(t) = \Lambda\left(\frac{t}{T}\right)$ first order reconstruction, requires 2 sample for each calculation

GRAPHICAL SUMMARY